Representation rings of $SL_2(\mathbb{F}_p)$ and S_p modulo projectives in characteristic p

Pavel Turek

Introduction

Let p be an odd prime and k an algebraically closed field of characteristic p. The representation theory of groups $SL_2(\mathbb{F}_p)$ and S_p over k is not too complicated thanks to the fact that both groups have a cyclic p-Sylow subgroup. In particular, if one works modulo projectives (that is, in the stable module category), the question of describing all modules is rather simple. Our tasks are to describe the indecomposable non-projective modules and the representation ring modulo projectives (that is, describe how to tensor the modules). Many of the preliminary results regarding $SL_2(\mathbb{F}_p)$ have been established for a long time and can be found for instance in [1].

Notation and results

Let E be the natural $kSL_2(\mathbb{F}_p)$-module. The non-projective irreducible modules are

$$\operatorname{Sym}^0 E \cong k, \operatorname{Sym}^1 E \cong E, \operatorname{Sym}^2 E, \operatorname{Sym}^3 E, \dots, \operatorname{Sym}^{p-2} E.$$

Similarly for S_p, let S denote the Specht module $S^{(p-1,1)}$ of kS_p and let D denote the simple module $D^{(p-1,1)}$ which is a quotient of S. Then the non-projective irreducible kS_p-modules are

$$\bigwedge^0 D \cong k, \bigwedge^1 D \cong D, \bigwedge^2 D, \bigwedge^3 D, \dots, \bigwedge^{p-2} D \cong \operatorname{sgn}.$$

The result which describes how to tensor irreducible $kSL_2(\mathbb{F}_p)$-modules is known as the Clebsch–Gordan rule. Using explicit maps, Green correspondence and inductive argument I have established an analogous result for S_p.

Clebsch–Gordan rule

Let $0 \le i \le j \le p-2$. Then module $\operatorname{Sym}^i E \otimes \operatorname{Sym}^j E$ equals up to projective

1. $\operatorname{Sym}^{j-i} E \oplus \operatorname{Sym}^{j-i+2} E \oplus \cdots \oplus \operatorname{Sym}^{i+j} E$ if $i+j < p-1$ where the exponents of the symmetric powers increase by two
2. $\operatorname{Sym}^{j-i} E \oplus \operatorname{Sym}^{j-i+2} E \oplus \cdots \oplus \operatorname{Sym}^{2p-4-i-j} E$ if $i+j \ge p-1$ where the exponents of the symmetric powers increase by two.

Moreover the same remains true if we replace E by D and the symmetric powers by the exterior powers.

We can introduce the following notation to obtain equivalent but more approachable formulas. Denote by $\mathcal{U}_i$ the element of the representation ring of $kSL_2(\mathbb{F}_p)$ modulo projectives corresponding to $\operatorname{Sym}^i E$ (for $0 \le i \le p-2$). We extend this definition to any integer i by setting $\mathcal{U}_{-1} = 0$ and requiring the sequence $(\mathcal{U}_i)_i$ to be $2p$-periodic and *antisymmetric along* $\mathcal{U}_{p-1}$, that is $\mathcal{U}_i + \mathcal{U}_{2p-2-i} = 0$ for any i. Then for any $i \ge 0$ and any integer j we have

$$\mathcal{U}_i \cdot \mathcal{U}_j = \mathcal{U}_{i-j} + \mathcal{U}_{i-j+2} + \cdots + \mathcal{U}_{i+j} \tag{1}$$

Note that the same can be done for S_p. We will denote the sequence by $(\mathcal{V}_i)_i$.

To complete our tasks it is now sufficient to use the Green correspondence and the Heller operator Ω. To do that we will view Ω as an operator acting on the representation ring modulo projectives. We start with S_p as the statement is simpler.

Representation rings modulo projectives

Each non-projective indecomposable kS_p-module can be written as $\Omega^i \mathcal{V}_{2j}$ where $i \in \mathbb{Z}/(2p-2)\mathbb{Z}$ and $0 \le j \le (p-3)/2$ and this is done in bijective fashion. The product $\Omega^{i_1}\mathcal{V}_{2j_1} \cdot \Omega^{i_2}\mathcal{V}_{2j_2}$ equals $\Omega^{i_1+i_2}(\mathcal{V}_{2j_1} \cdot \mathcal{V}_{2j_2})$ and so (1) can be used. For $SL_2(\mathbb{F}_p)$ the form becomes $\mathcal{U}_{p-2}^{\epsilon} \cdot \Omega^i \mathcal{U}_{2j}$ where $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ and $i \in \mathbb{Z}/(p-1)\mathbb{Z}$. Multiplication works analogously, that is we separately add ϵ's and i's and use (1).

Diagrammatic description

S^*			0			$-S^*$
$\Lambda^2 S^*$			0			$-\Lambda^2 S^*$
$\Lambda^3 S^*$		A	0			$-\Lambda^3 S^*$
$\Lambda^4 S^*$			0			$-\Lambda^4 S^*$
$\Lambda^5 S^*$			0			$-\Lambda^5 S^*$
sgn	$\Lambda^3 D$	D	0	$-D$	$-\Lambda^3 D$	$-$sgn
$\Lambda^5 S$			0			$-\Lambda^5 S$
$\Lambda^4 S$			0			$-\Lambda^4 S$
$\Lambda^3 S$			0			$-\Lambda^3 S$
$\Lambda^2 S$		B	0			$-\Lambda^2 S$
S			0			$-S$
k	$\Lambda^2 D$	$\Lambda^4 D$	0	$-\Lambda^4 D$	$-\Lambda^2 D$	$-k$

Table 1: The diagrammatic description of the representation ring of kS_p in characteristic p modulo projectives when $p = 7$. The product of A and B is represented by boxes with magenta borders, where the darker version represents the modules left after cancellations.

			0			
			0			
			0			
			0			
			0			
k	$S^2 E$	$S^4 E$	0	$-S^4 E$	$-S^2 E$	$-k$

Block $\epsilon = 0$

			0			
			0			
		A	0			
			0			
	B		0			
$S^5 E$	$S^3 E$	E	0	$-E$	$-S^3 E$	$-S^5 E$

Block $\epsilon = 1$

Table 2: The diagrammatic description of the representation ring of $kSL_2(\mathbb{F}_p)$ in characteristic p modulo projectives when $p = 7$. The product of A and B is represented by boxes with magenta borders, where the darker version represents the modules left after cancellations. Note that the notation S^i is used for the i-th symmetric power.

The non-projective indecomposable modules of kS_p are represented by green boxes in Table 1 and similarly for $kSL_2(\mathbb{F}_p)$ in Table 2. Note that the darker green is used for the irreducible modules. The middle white boxes represent zero modules and red boxes represent negative modules which are obtained by reflecting along the column of zero modules. In Table 1 the module $\Omega^i \mathcal{V}_{2j}$ corresponds to the box in row i (counted from below when the bottom row is 0) and column j (counted from left starting with 0). So one can think of the columns as a sequence $(\mathcal{V}_{2j})$, where our table only pictures one period of this sequence.

The advantage of this table is that one can easily perform multiplication. If one starts with two green boxes A and B, to obtain the product one needs to add up some boxes which are in the row indexed by the sum of indices of rows of A and B. In our example the index is $2 + 9 = 11$. The columns of the boxes are recovered from (1) as follows. If A is in the column j one needs to add up the boxes in columns which are at most j away from the column of B (note that one needs to keep in mind that the table has infinitely many columns, this is just one period). In our case $j = 2$ and so one needs to add up 5 boxes. These are given by squares with magenta borders. After forgetting the zero module and cancelling the green module in column 2 with the red negative module one ends up with the sum of the modules in columns 0 and 1. The Table 2 is similar but we have two smaller tables corresponding to different values of ϵ.

Algebraic description

Let R_D denote the representation ring of kS_p modulo projectives. Similarly, use R_E for the group $SL_2(\mathbb{F}_p)$. One can describe the rings using a primitive p'th root of unity which we denote by ζ_p. This idea has appeared in [2] in the context of the stable module category of the cyclic group $\mathbb{Z}/p\mathbb{Z}$. Define $\alpha = \zeta_p + \zeta_p^{-1} \in \mathbb{Z}[\zeta_p]$.

Isomorphisms

The map $\Phi_D : \mathbb{Z}[\alpha][X]/(X^{2p-2} - 1) \to R_D$ given by $\alpha \mapsto \mathcal{V}_2 - \mathcal{V}_0$ and $X \mapsto \Omega\mathcal{V}_0$ is an isomorphism of rings. The analogous isomorphism for $SL_2(\mathbb{F}_p)$ is given by $\Phi_E : \mathbb{Z}[\alpha][X, Y]/(X^{p-1} - 1, Y^2 - 1) \to R_E$ where $\alpha \mapsto \mathcal{U}_2 - \mathcal{U}_0$, $X \mapsto \Omega\mathcal{U}_0$ and $Y \mapsto \mathcal{U}_{p-2}$.

Further directions and references

An interesting question concerns applying Schur functors to our modules. For example, the symmetric and exterior powers are studied in [1] and the sequential papers.

References

[1] F. M. Kouwenhoven, "The λstructure of the green ring of GL(2, Fp) in characteristic p," *Communications in Algebra*, vol. 18, no. 6, pp. 1645–1671, 1990.

[2] G. Almkvist, "Representations of Z/pZ in characteristic p and reciprocity theorems," *Journal of Algebra*, vol. 68, no. 1, pp. 1–27, 1981. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0021869381902817